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FOURIER SERIES AND DELTA-SUBHARMONIC FUNCTIONS OF ZERO-TYPE IN A HALF-PLANE

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In terms of Fourier coefficients and associated complete measures a class of just δ-subharmonic functions in a half-plane of a zero type is described.

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В терминах коэффициентов Фурье и ассоциированных полных мер описан класс истинно δ-субгармонических в верхней полуплоскости функций нулевого типа.

Introduction. In the 60s several American authors (Rubel, Taylor [1], Miles [2], Shea and others) started to use the Fourier series method for the study of the properties of entire and meromorphic functions. An advantage of this method is its suitability for the investigation of functions of fairly irregular growth at infinity and functions of infinite order. Later important results in this direction were obtained by Kondratyuk [3], [4], [5], who generalized the Levin-Pflüger theory of entire functions of completely regular growth to meromorphic functions of arbitrary γ-type. In paper [6], the results of Rubel, Taylor, Miles were extended to delta-subharmonic functions in a half-plane. In the present paper we extend some of the results of paper [6] to functions of zero-type in a half-plane.

Let $J\delta$ be the class of just δ-subharmonic functions, and $J\delta(0)$ be the class of just δ-subharmonic functions of zero-type (we present the definitions of this class below) in the upper half-plane.

Theorem 1. *Let $v \in J\delta$. Then the following two properties are equivalent: (1) $v \in J\delta(0)$; (2) the measure $\lambda_+(v)$ (or $\lambda_-(v)$) has finite zero-density and*

$$|c_k(r, v)| \le A, \quad k \in \mathbb{N},$$

for some positive A and all $r > 0$.

Here $\lambda(v)=\lambda_+(v)-\lambda_-(v)$ is the complete measure corresponding to the function v and $c_k(r, v)$ are the Fourier coefficients of v. A result similar one of Miles's [2] also holds for the class $J\delta(0)$

$$J\delta(0) = JS(0) - JS(0),$$

where $JS(0)$ is the class of just subharmonic functions of zero-type in the upper half-plane.

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1. Classes of functions in $\mathbb{C}_+$. In this paper we use terminology from [7], [8]. Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ be the upper half-plane. We denote by $C(a, r)$ the open disk of radius r with center at a, and by Ω_+ the intersection of a set Ω with the half-plane $\mathbb{C}_+$: $\Omega_+ = \Omega \cap \mathbb{C}_+$. A subharmonic function v in $\mathbb{C}_+$ is said to be just subharmonic if $\limsup_{z\to t} v(z) \le 0$ for each $t \in \mathbb{R}$. The class of just subharmonic functions in $\mathbb{C}_+$ will be denoted by JS. Let SK be the class of subharmonic functions in $\mathbb{C}_+$ possessing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. Functions in SK have the following properties [7]:

(a) $v(z)$ has non-tangential limits $v(t)$ almost everywhere on the real axis and $v(t) \in L^1_{\mathrm{loc}}(-\infty, \infty)$;

(b) there exists a charge ν on the real axis such that
$$\lim_{y\to+0} \int_a^b v(t + iy)\, dt = \nu([a, b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\}),$$
the measure ν is called the boundary measure of v;

(c) $d\nu(t) = v(t)\, dt + d\sigma(t)$, where σ is a singular measure with respect to the Lebesgue measure.

Following [7] we define for a function $v \in SK$ the corresponding complete measure λ by the formula
$$\lambda(K) = 2\pi \int_{\mathbb{C}_+\cap K} \operatorname{Im} \zeta\, d\mu(\zeta) - \nu(K),$$
where μ is the Riesz measure of v. The measure λ has the following properties:

(1) λ is the finite measure on each compact subset K of $\mathbb{C}$;

(2) λ is a positive measure outside $\mathbb{R}$;

(3) λ vanishes in the half-plane $\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}$.

Conversely, if λ is a measure with properties (1) – (3), then there exists a function $v \in SK$ with complete measure λ. The collection of properties (1) – (3) will be denoted by $\{\mathbf{G}\}$ in what follows; if, in addition, λ is also a non-negative measure in $\mathbb{R}$, then we denote the corresponding collection by $\{\mathbf{G}^+\}$.

If D is bounded subdomain of $\mathbb{C}_+$ and $D_1 = D \cup (\partial D \cap \mathbb{R})$, $v \in SK$, $z \in D$, then
$$v(z) = \frac{1}{2\pi} \iint_{D_1} \frac{1}{\operatorname{Im} \zeta} \ln \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| d\lambda(\zeta) + h(z),$$
where h is a harmonic function in D, and if $[a, b] \subset \{\mathbb{R} \cap \partial D\}$, then h admits a continuous extension by zero to (a, b); we assume that $\frac{1}{\operatorname{Im} \zeta} \ln \left| \frac{z - \zeta}{z - \bar{\zeta}} \right|$ is extended to the real axis by continuity. The complete measure λ determines a function $v \in SK$ to the same extent as the Riesz measure μ determines a subharmonic function in $\mathbb{C}$. More precisely, if $v_1, v_2 \in SK$ are two functions with complete measure λ, then there exists a real entire function g such that $v_2(z) - v_1(z) = \operatorname{Im} g(z)$, $z \in \mathbb{C}_+$.

The following result holds ([7]).

Proposition 1. $JS \subset SK$.

The complete measure of a function $v \in JS$ is a positive measure, which explains the notion "just subharmonic function".

Let us now introduce the class of just δ-subharmonic function $J\delta = JS - JS$.

Proposition 2. $J\delta = SK - SK$.

For a fixed measure λ let

$$d\lambda_m(\zeta)=\frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}\,d\lambda(\zeta),\quad \zeta=\tau e^{i\varphi},\quad \lambda_m(r)=\lambda_m\big(\overline{C(0,r)}\big)\,,$$

where $\sin m\varphi/\sin\varphi$ is defined for $\varphi\in\{0,\ \pi\}$ by continuity.

The next relation is Carleman's formula in Grishin's notation

$$\frac{1}{r^k}\int\limits_0^\pi v(re^{i\varphi})\sin k\varphi\,d\varphi=\int\limits_{r_0}^r\frac{\lambda_k(t)}{t^{2k+1}}\,dt+\frac{1}{r_0^k}\int\limits_0^\pi v(r_0e^{i\varphi})\sin k\varphi\,d\varphi\,, \tag{1}$$

in particular, for $k=1$ we have

$$\frac{1}{r}\int\limits_0^\pi v(re^{i\varphi})\sin\varphi\,d\varphi=\int\limits_{r_0}^r\frac{\lambda(t)}{t^3}\,dt+\frac{1}{r_0}\int\limits_0^\pi v(r_0e^{i\varphi})\sin\varphi\,d\varphi \tag{2}$$

for all $r>r_0$. Note also another inequality, which is useful in what follows

$$|\lambda_m(r)|=\left|\iint_{\overline{C(0,r)}}d\lambda_m(\zeta)\right|=\left|\iint_{\overline{C(0,r)}}\frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}\,d\lambda(\zeta)\right|\le$$

$$\le m\iint_{\overline{C(0,r)}}\tau^{m-1}\,d|\lambda|(\zeta)\le mr^{m-1}|\lambda|(r)\,. \tag{3}$$

Let $D_+(R_1,R_2)=\overline{C_+(0,R_2)\backslash C_+(0,R_1)}$, $R_1<R_2$. Functions $v\in J\delta$ have representations in the half-annulus $z\in D_+(R_1,R_2)$

$$v(z)=-\frac{1}{2\pi}\iint_{D_+(R_1,R_2)}K(z,\zeta)\,d\lambda(\zeta)+\frac{R_2}{2\pi}\int\limits_0^\pi\frac{\partial G(z,R_2e^{i\varphi})}{\partial n}v\left(R_2e^{i\varphi}\right)\,d\varphi+$$

$$+\frac{R_1}{2\pi}\int\limits_0^\pi\frac{\partial G(z,R_1e^{i\varphi})}{\partial n}v(R_1e^{i\varphi})\,d\varphi\,, \tag{4}$$

and in the half-disk $z\in C_+(0,R)$

$$v(z)=-\frac{1}{2\pi}\iint_{\overline{C_+(0,R)}}K(z,\zeta)\,d\lambda(\zeta)+\frac{R}{2\pi}\int\limits_0^\pi\frac{\partial G(z,Re^{i\varphi})}{\partial n}v(Re^{i\varphi})\,d\varphi\,,$$

where $G(z,\zeta)$ is the Greens function of the half-annulus (the half-disk), $\dfrac{\partial G}{\partial n}$ is its derivative in the inward normal direction, and the function $K(z,\zeta)=\dfrac{1}{\operatorname{Im}\zeta}G(z,\zeta)$, $\zeta\in D_+(R_1,R_2)$ (and $z\in C_+(0,R)$), is extended by continuity to the points on the real axis with $R_1\le|t|\le R_2$.

Using the theory of elliptic functions (see, for instance, [9], Chapter VIII) one can obtain expansions of the kernel in formula (4) for $R_1=qR$, $R_2=R/q$, $q\in(0,1)$, $z=re^{i\theta}$, $\zeta=\tau e^{i\varphi}$ and $qR\le\tau<r<\frac{1}{q}R$

$$G(z,\zeta)=2\sum_{m=1}^\infty\frac{1}{m(1-q^{4m})}\left(\frac{\tau}{r}\right)^m\left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\left(1-\frac{q^{2m}R^{2m}}{\tau^{2m}}\right)\sin m\theta\sin m\varphi, \tag{5}$$

for $qR \le r < \tau \le \frac{1}{q}R$

$$G(z,\zeta) = 2\sum_{m=1}^{\infty} \frac{1}{m(1-q^{4m})}\left(\frac{r}{\tau}\right)^m \left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\left(1-\frac{q^{2m}\tau^{2m}}{R^{2m}}\right)\sin m\theta \sin m\varphi, \tag{6}$$

for $qR \le |t| \le r \le R/q$

$$\frac{\partial G(z,t)}{\partial n} = \frac{2}{t}\sum_{m=1}^{\infty} \frac{1}{m(1-q^{4m})}\left(\frac{t}{r}\right)^m \left(1-\frac{q^{2m}R^{2m}}{t^{2m}}\right)\left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\sin m\theta\,, \tag{7}$$

for $qR \le r \le |t| \le R/q$

$$\frac{\partial G(z,t)}{\partial n} = \frac{2}{t}\sum_{m=1}^{\infty} \frac{1}{m(1-q^{4m})}\left(\frac{r}{t}\right)^m \left(1-\frac{q^{2m}t^{2m}}{R^{2m}}\right)\left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\sin m\theta\,, \tag{8}$$

$$\frac{\partial G\big(z,qRe^{i\varphi}\big)}{\partial n} = \frac{4}{qR}\sum_{m=1}^{\infty} \frac{1}{1-q^{4m}}\left(\frac{qR}{r}\right)^m \left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\sin m\theta \sin m\varphi\,, \tag{9}$$

$$\frac{\partial G\big(z,\frac{1}{q}Re^{i\varphi}\big)}{\partial n} = \frac{4q}{R}\sum_{m=1}^{\infty} \frac{1}{1-q^{4m}}\left(\frac{qr}{R}\right)^m \left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\sin m\theta \sin m\varphi\,. \tag{10}$$

2. Fourier coefficients of functions of class $J\delta$. The Fourier coefficients of a function $v \in J\delta$ are defined as usual ([10])

$$c_k(r,v) = \frac{2}{\pi}\int_0^{\pi} v(re^{i\theta})\sin k\theta\, d\theta, \quad k \in \mathbb{N}\,.$$

From (1) we obtain the following expressions for the Fourier coefficients for $r > r_0$

$$c_k(r,v) = \alpha_k r^k + \frac{2r^k}{\pi}\int_{r_0}^{r} \frac{\lambda_k(t)}{t^{2k+1}}\,dt, \quad k \in \mathbb{N}\,, \quad \alpha_k = r_0^{-k}c_k(r_0,v). \tag{11}$$

Applying the formula of integration by parts to the integral in (11), we obtain

$$c_k(r,v) = \alpha_k r^k + \frac{r^k}{\pi k r_0^{2k}}\iint_{\overline{C_+(0,r_0)}} \frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^k\,d\lambda(\zeta) +$$

$$+\frac{r^k}{\pi k}\iint_{D_+(r_0,r)} \frac{\sin k\varphi}{\tau^k \operatorname{Im}\zeta}\,d\lambda(\zeta) - \frac{1}{r^k\pi k}\iint_{\overline{C_+(0,r)}} \frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^k\,d\lambda(\zeta)\,, \quad \zeta = \tau e^{i\varphi}. \tag{12}$$

Proposition 3. *The Fourier coefficients $c_k(r,v)$ of a function $v \in J\delta$ are continuous functions of r.*

This follows from the fact that the right-hand sides of relations (12) are continuous functions of r.

3. Subharmonic and δ-subharmonic functions of zero-type. For $v \in J\delta$ let $v = v_+ - v_-$, let λ be the complete measure of v and let $\lambda = \lambda_+ - \lambda_-$ be the Jordan decomposition of λ (note that λ_- is not the complete measure of v_-). We set

$$m(r,v) := \frac{1}{r}\int_0^{\pi} v_+(re^{i\varphi})\sin\varphi\,d\varphi, \quad N(r_1,r_2,v) := \int_{r_1}^{r_2} \frac{\lambda_-(t)}{t^3}\,dt \quad (0 < r_1 < r_2)\,,$$

$$T(r_1,r_2,v) := m(r_2,v) + N(r_1,r_2,v) + m(r_1,-v)\,.$$

In this notation Carleman's formula (2) can be written as follows:

$$T(r_1, r_2, v) = T(r_1, r_2, -v)\,. \tag{13}$$

Definition 1. A function $v \in J\delta$ is called a *function of zero-type* if there exists a positive constant A such that

$$T(r_1, r_2, v) \le \frac{A}{r_1} \quad \text{for all } r_2 > r_1 > 0.$$

We denote the corresponding class of δ-subharmonic functions of zero-type by $J\delta(0)$. Let $JS(0)$ be the class of just subharmonic functions of zero-type.

If $v \in J\delta(0)$, then it follows from (13) that

$$\lim_{r\to\infty} m(r, v) = 0$$

and

$$(\forall\ r > 0): \quad N(r, v) := N(r, \infty, v) < \infty.$$

For a function $v \in J\delta(0)$ we set

$$T(r, v) := m(r, v) + N(r, v)\,.$$

Definition 2. A function $v \in J\delta$ is called a *function of zero-type* if there exists a positive constant A such that

$$(\forall\ r > 0): \quad T(r, v) \le \frac{A}{r}.$$

It is clear that Definition 2 and Definition 1 are equivalent. In this notation Carleman's formula (2) can be written as follows:

$$T(r, v) = T(r, -v)\,. \tag{14}$$

Lemma 1. *The class $J\delta(0)$ is a real linear space and $JS(0)$ is a real cone.*

This is a consequence of (13) and the inequality $T\left(r, \sum v_j\right) \le \sum T(r, v_j)$.

Definition 3. A positive measure λ has *zero-density* if there exists a positive constant A such that

$$(\forall\ r > 0): \ N(r, \lambda) := \int_r^\infty \frac{\lambda(t)}{t^3}\, dt \le \frac{A}{r}$$

.

Definition 4. A positive measure λ is called a *measure of zero-type* if there exists a positive constant A such that

$$(\forall\ r > 0): \ \lambda(r) \le Ar\,. \tag{15}$$

Lemma 2. *If λ is a measure of zero-density, then it is a measure of zero-type.*

The proof is provided by the inequalities $N(r, \lambda) = \displaystyle\int_r^\infty \frac{\lambda(t)}{t^3}\, dt \ge \int_r^{er} \frac{\lambda(t)}{t^3}\, dt \ge \frac{\lambda(r)}{e^2 r^2}$.

4. Proof of Main Theorem. Let $v \in J\delta(0)$. Note first that each of the measures $\lambda_+(v)$ and $\lambda_-(v)$ has zero-density. The measure $\lambda_-(v)$ has zero-density by the definition of the class $J\delta(0)$. The fact that $\lambda_+(v)$ has zero-density is a consequence of (14). The same formula yields

$$\int_0^\pi \left| v(re^{i\varphi}) \right| \sin\varphi\, d\varphi \le A\,. \tag{16}$$

Note also that the measure $|\lambda| = \lambda_+ + \lambda_-$ has zero-density and therefore satisfies inequality (15). From (16) we obtain

$$|c_k(r, v)| \le Ak\,. \tag{17}$$

Formula (1) yields $c_k(r,v)=\dfrac{1}{2^k}c_k(2r,v)-\dfrac{2r^k}{\pi}\displaystyle\int_r^{2r}\frac{\lambda_k(t)}{t^{2k+1}}\,dt$, which, in view of (3), (15) and (17), gives us the inequality $|c_k(r,v)|\le\dfrac{Ak}{2^k}+\dfrac{2A}{\pi}$. This completes the proof of the implication (1) $\Longrightarrow$ (2).

Assume now that condition (2) in the theorem holds. Then it follows by the inequality $|c_1(r,v)|\le A$ and formula (2) that if one of measure $\lambda_+(v)$ and $\lambda_-(v)$ has zero-density, then the other measure also has zero-density, and therefore $|\lambda|$ has zero-density. We can now find an estimate of $v_+(z)$ using formula (4) with $R_1=r/2$, $R_2=2r$. By considering the expansion (9), (10) in Fourier series for $q=1/2$ and $R=r=|z|$ we obtain

$$\Big|\frac{r}{\pi}\int_0^\pi\frac{\partial G\left(z,2re^{i\varphi}\right)}{\partial n}v\left(2re^{i\varphi}\right)\,d\varphi+\frac{r}{4\pi}\int_0^\pi\frac{\partial G\left(z,\frac12re^{i\varphi}\right)}{\partial n}v\left(\frac12re^{i\varphi}\right)\,d\varphi\Big|\le$$
$$\le\sum_{m=1}^\infty\frac{1}{2^m}\frac{4^m}{1+4^m}\Big[2|c_m(2r,v)|+8\big|c_m\big(\frac{r}{2},v\big)\big|\Big]\le A\quad\text{for some } A>0.$$

This inequality and formula (4) yield $v_+(z)\le\dfrac{1}{2\pi}\displaystyle\iint_{D_+\left(\frac12r,2r\right)}K(z,\zeta)\,d\lambda_-(\zeta)+A$. Now, using the orthogonality of system of polynomials $\{\sin k\theta\}$, $k\in\{1,2,...\}$, on the interval $[0,\pi]$ and formulae (5)–(8) we obtain

$$\int_0^\pi v_+\Big(re^{i\theta}\Big)\sin\theta\,d\theta\le\frac{1}{2\pi}\int_0^\pi\left\{\left[\iint_{D_+(r/2,r)}+\iint_{D_+(r,2r)}\right]K(z,\zeta)\,d\lambda_-(\zeta)\right\}\sin\theta\,d\theta+2A\le$$
$$\le\frac12\iint_{D_+(r/2,r)}\frac{\sin\varphi}{\operatorname{Im}\zeta}\frac{\tau}{r}\frac{4}{5}\left(1-\frac{r^2}{4\tau^2}\right)d\lambda_-(\zeta)+\frac12\iint_{D_+(r,2r)}\frac{\sin\varphi}{\operatorname{Im}\zeta}\frac{r}{\tau}\frac{4}{5}\left(1-\frac{\tau^2}{4r^2}\right)d\lambda_-(\zeta)+$$
$$+2A\le\frac{2}{5r}\int_{r/2}^r\left(1-\frac{r^2}{4\tau^2}\right)d\lambda_-(\tau)+\frac{2}{5r}\int_{r/2}^r\left(1-\frac{\tau^2}{4r^2}\right)d\lambda_-(\tau)+2A\le\frac{2\lambda_-(2r)}{5r}+2A\,.$$

Since the measure λ_- has zero-density, it is a measure of zero-type according to the Lemma 2. Hence, the right-hand side of the last inequality is bounded. This yields $m(r,v)\le C/r$, C is a constant. Together with the inequality $N(r,v)\le A/r$, A is a constant, this gives $v\in J\delta(0)$.

Theorem 2. *Let $v\in JS$. Then the following properties are equivalent:* 1^0. $v\in JS(0)$; 2^0. $|c_k(r,v)|\le A,\quad k\in\mathbb{N}$, *for some positive A and for all $r>0$.*

This is an immediate consequence of Theorem 1 because the measure λ_- vanishes for functions in the class JS.

In addition to Theorem 1 we claim that property (1) does not yield the following refinement of 2^0 $|c_k(r)|\le\varepsilon_k$ with $\varepsilon_k\to0$ as $k\to\infty$). It can be seen in the example below

Example. Consider a harmonic and non-positive function in $\mathbb{C}_+$

$$v(z)=\sum_{k=-\infty}^{+\infty}\operatorname{Im}\frac{2^k}{z-2^k}\,.$$

The function v satisfies the relation $v(2z)=v(z)$. Hence $C_1\le\int\limits_0^\pi|v(re^{i\varphi})|\sin\varphi\,d\varphi\le C_2$, where $C_1=\inf\big\{-\int_0^\pi v(re^{i\theta})\sin\theta d\theta\,:\,r\in[1;2]\big\}$, $C_2=\sup\big\{-\int_0^\pi v(re^{i\theta})\sin\theta\,d\theta\,:\,r\in[1;2]\big\}$. Let $z=re^{i\theta}$, $2^{k_0}<r<2^{k_0+1}$. Using the expansion

$$v\big(re^{i\theta}\big)=-\sum_{2^k<r}\sum_{m=0}^\infty\frac{2^{mk}\sin(m+1)\theta}{r^{m+1}}-\sum_{2^k>r}\sum_{m=1}^\infty\frac{r^m\sin m\theta}{2^{mk}}$$

we can verify by the direct calculation that

$$c_n(r,v) = -\frac{1}{r^n}\sum_{2^k<r} 2^{kn} - r^n \sum_{2^k>r} 2^{-kn} = -\frac{1}{r^n}\sum_{k=-\infty}^{k_0} 2^{kn} - r^n \sum_{k=k_0+1}^{\infty} 2^{-kn} =$$
$$= -\frac{1}{r^n}\frac{2^{k_0 n}}{1-2^{-n}} - r^n \frac{2^{-(k_0+1)n}}{1-2^{-n}}.$$

By Proposition 3 these relations hold also for $r = r_0 = 2^{k_0}$

$$c_n(r_0,v) = -\frac{1}{2^{k_0}}\frac{2^{nk_0}}{1-2^{-n}} - 2^{k_0}\frac{2^{-n(k_0+1)}}{1-2^{-n}} = -\frac{1}{1-2^{-n}} - \frac{1}{2^n(1-2^{-n})}.$$

Hence $|c_n(r_0,v)| \geq 1,\quad n \in \mathbb{N}$. Since $r_0 = 2^{k_0}$ can be taken arbitrarily large, the last inequality shows that Theorem 1 cannot be refined.

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